The Return of the Universal ∂-Functors

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Abstract

It is known that the universal higher topos approach, e.g. using Eilenberg-MacLane spaces, to cohomology, lacks the computational accessability of the more classical definitions. A similar gap exists, if homotopy type theory is used as a basis for synthetic mathematics, for example for synthetic differential geometry or the more recently extended synthetic algebraic geometry. In synthetic mathematics, which we consider to take place internally in some particular toposes, the gap seems to be even wider, since the middle ground of the derived functors of homological algebra does not support one of the standard techniques: injective resolutions use the axiom of choice and there is no hope of having an internal construction. This work shows that there is a way around this gap - at least in some special cases. From what is shown in special cases, it is plausibel to hope for a general theory, where the connection between cohomology via higher types is connected to homological algebra by a choice principle related to the topology of a grothendieck topos.

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1 Idea

In pure mathematics, interesting information about objects X can often be captured by a sequence of abelian groups, called the cohomology of X. Besides the

group structure, the construction is functorial and there is usually also a ring structure on the totality of the groups and lots of properties hold. There is a second parameter, the coefficients. The default in homotopy theory is usually the group $(\mathbb{Z}, +)$, while it can also be a sheaf of special functions on a space in algebraic or differential geometry.
In these notes, we will use a somewhat overly abstract definition of cohomology, where the coefficients can be a spectrum varying over a space. This is a convenient way to set everything up and contains the case we are really interested in: cohomology with coefficients in an abelian group varying over a space.
For an object X in a topos, this would include the case of a sheaf of abelian groups on A, which received a lot of attention historically. In the classical way to develope a theory of cohomology of sheaves, injective resolutions are used in basic definitions and more advanced calculations. These notes are about a simple test case, which goes a small step in the direction of making similar computations internally in homotopy type theory.
We do not believe, that there is a genral way to construct injective resolutions internally. Instead, we will aim at mimicing, what is called an *effacement* in Grothendieck's *Tohoku*-Paper ([4]). Roughly, an effacement of an object A of an abelian category, is a monomorphism $A \to M$, that will be zero on cohomology groups, starting in degree one. In contrast to injective resolutions, effacements are specific to a single functor and not something which might exist in general in the domain of a functor.
Instead of an effacement, that erases all cohomology on a space, we will introduce a simple construction of a monomorphism, which erases one specific element of a specific cohomology group for some special situations – along the same lines as [1]. This turns out to be enough to repeat one calculation, which establishes that cohomology has a universal property – that of a so called *universal ∂-Functor*. Which is almost the same statement as [1][Propposition 4.2], but in a different context.

Acknowledgements

Urs Schreiber helped me a lot in the more distant past to understand how cohomology can be defined and used in toposes.
There is a prehistory of this article which started in Augsburg, where I worked on related problems with Lukas Stoll and Ingo Blechschmidt. Our discussions, also those with Marc Nieper-WiSSkirchen, were important to my understanding of the subject.
Later, in Gothenburg, this work was supported by discussions with Thierry Coquand, David Wärn and Christian Sattler. At the same time, Tobias Columbus helped a lot by urging me to find a universal property of the resolutions in **??**. While this is not how the material is organized now, it was an important step along the way.
Finally, during and after a presentation, various remarks and questions helped to improve the article. Ingo Blechschmidt and Lukas Stoll found gaps in my notes

and had the idea to call the resolutions in the article local.
And, most crucially, about ten years ago, Michael Fütterer gave a talk about group cohomology in our student organized seminar in Karlsruhe, where he thouroughly explained how delta-functors are used. Without those memories in the back of my mind, there would have been no basis for the ideas in this article.

2 Spectra and Cohomology

A *spectrum* is roughly an infinit delooping of a (pointed) space. An n-th delooping of a pointed space A which is also $(n-1)$-connected is unique and usually written as $B^n A$ or $K(A,n)$ and called an *Eilenberg-MacLane space*. We will just write A_n for an n-th delooping in this article.
It is known, that in HoTT, a (0-truncated) abelian group can be delooped arbitrarily often ([5]).
Contents of this section are from Mike Shulman's posts on the HoTT-Blog about cohomology, Floris van Doorn's thesis ([6])[section 5.3] and common knowledge in the field that is not written up.
Suppose we have a pointed type A with delooping A_k for any $k : \mathbb{N}$. Then, analogous to the definition of the k-th homotopy group

$$\pi_k :\equiv \|\Omega^k A\|_0$$

one could define homotopy groups of *negative* degree $-k$ by:

$$\pi_{-k} :\equiv \|A_k\|_0$$

Note that these will be trivial for any Eilenberg-MacLane spectrum, since for those, A_{k+1} is k-connected for $k : \mathbb{N}$. In general, spectra with trivial homotopy groups in negative degree are called *connective*. The result in this article is concerned with Eilenberg-MacLane spectra.
We will use spectra varying over a space as coefficints for cohomology, which corresponds to the classical concept of parametrized spectra. We fix our terminology in the following definition.

Definition 2.1

(a) A *spectrum* is a sequence of pointed types $(A_k)_{k:\mathbb{N}}$, together with pointed equivalences $A_k \simeq \Omega A_{k+1}$.

(b) A spectrum $(A_k)_{k:\mathbb{N}}$ is *connective* or an *abelian ∞-group*, if $\|A_{k+1}\|_0 \simeq 1$ for all $k : \mathbb{N}$.

(c) Let X be a type. A *parametrized spectrum* over X, is a dependent function, which assigns to any $x : X$, a spectrum $(A_{x,k})_{k:\mathbb{N}}$. For brevity, We will call a parametrized spectrum $A \equiv x \mapsto (A_{x,k})_{k:\mathbb{N}}$ over X just *spectrum over X*.

(d) A *morphism of spectra* A, A' over X, is given by a sequence of pointed maps $f_{x,k} : A_{x,k} \to A'_{x,k}$ for any $x : X$, such that $\Omega f_{x,k+1} = f_{x,k}$ (using the pointed equivalences).

The connective spectra form a nice "subcategory": We will need the following (coreflective) construction that turns a spectrum into a connective spectrum. See definition 6.9 for the definition of the k-connected cover "$D^k_X d$".

Definition 2.2
For a spectrum A, the following construction is called the *connective cover*:

$$\hat{A} :\equiv k \mapsto D^{k-1}_{A,k}$$

There is also a sequence of pointed maps $\varphi_k : \hat{A}_k \to A_k$, given by the projection from the connected covers.

The overall purpose of these notes, is to provide a general result about the calculation of cohomology of types with coefficients in Eilenberg-MacLane spectra over them. However, using a general spectrum seemed to make things a bit clearer. The central result is a theorem, which establishes a universal property of this cohomology construction. We will postpone the universal property and start with the basic concepts and continue with some constructions we will need.

Definition 2.3
The k-th cohomology group of X with coefficients in A is the following:

$$H^k(X, A) :\equiv \|(x : X) \to A_{x,k}\|_0$$

An important notion in abelian categories, is that of short exact sequences. And it is important to us here, since for every short exact sequence (somewhere), there should be an induced long exact sequence on cohomology groups. The cokernel of an exact sequence, corresponds to a cofiber of a map of spectra. This is surprisingly pleasant to construct and work with and I have to thank David Jaz Myers for explaining this to me:

Definition 2.4
Let $f : A \to A'$ be a map of spectra.

(a) The *cofiber* of f is given by the spectrum

$$C_{f,k} :\equiv \mathrm{fib}_{f_{k+1}}$$

together with the map $c : A' \to C_f$, where c_k is induced in the following diagram of pullback-squares:

$$\begin{array}{ccccc}
A_k & \xrightarrow{f_k} & A'_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & C_{f,k} & \longrightarrow & A_{k+1} \\
 & & \downarrow & & \downarrow \\
 & & 1 & \longrightarrow & A'_{k+1}
\end{array}$$

(b) The *fiber* of f is given by the spectrum

$$\mathrm{fib}_{f,k} :\equiv \mathrm{fib}_{f_k}$$

Note that $f : A \to A'$ is always the fiber of its cofiber and conversely, $f : A \to A'$ is always the cofiber of its fiber, which is very different from the situation in a general abelian category, where for example not every map is the kernel of its cokernel.
For the following definition, we just pick one of the two possibilities to name it – the "cofiber" is a bit better at reminding us, that we can do more with it than what we are used to from homotopy fibers. For Eilenberg-MacLane spectra, this notion coincides with that of a short exact sequence of the underlying abelian groups.

Definition 2.5
A sequence of morphisms of spectra over X

$$A \xrightarrow{f} A' \xrightarrow{g} A''$$

is a *cofiber sequence*, if the following equivalent statements hold:

(a) g_x is the cofiber of f_x for all $x : X$

(b) f_x is the fiber of g_x for all $x : X$

(c) $f_{x,k}$ is the fiber of $g_{x,k}$ for all $x : X$ and $k : \mathbb{N}$

If all spectra involved are Eilenberg-MacLane spectra, we call the sequence *exact*, and vice versa, if we speak of a *short exact sequence* of spectra (over X), we assume all spectra involved are Eilenberg-MacLane and we have a cofiber sequence.

Lemma 2.6
If $A \to A' \to A''$ is a cofiber sequence, then the induced square:

$$\begin{array}{ccc} \prod_{x:X} A_{x,k} & \longrightarrow & \prod_{x:X} A'_{x,k} \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \prod_{x:X} A''_{x,k} \end{array}$$

is a pullback square for all $k : \mathbb{N}$.

Proof $\prod$ maps families of pullback squares to a pullback square.

This is just tailored to prove the following proposition:

Proposition 2.7
For any cofiber sequence

$$A \to A' \to A''$$

of spectra over X, there is a long exact sequence of cohomology groups:

$$
\begin{array}{ccccc}
 & & \dots & \longrightarrow & H^{n-1}(X,A'') \\
 & \swarrow & & & \\
H^n(X,A) & \longrightarrow & H^n(X,A') & \longrightarrow & H^n(X,A'') \\
 & \swarrow & & & \\
H^{n+1}(X,A) & \longrightarrow & H^{n+1}(X,A') & \longrightarrow & \dots
\end{array}
$$

Proof Apply homotopy fiber sequence to last proposition for all $k : \mathbb{N}$.

3 Generic Resolution of cohomology classes

Contents of this section should be new.
Terminology is somewhat made up and not really commonly used. This section produces local resolutions, which allow a proof of universality of H^1 as a ∂-functor. It doesn't seem to be possible, to extend this construction for higher degrees.
We will start by explaining how the construction of the *local resolutions* in this section was derived. For the first iteration, only elements of the first cohomology group $H^1(X, A)$ were considered. In this case, an element $T : H^1(X, A)$ can be represented by an A-torsor, or more precisely, T_x is an A_x-torsor for any $x : X$. Let us relax the usual notion of fibre bundle a bit, to also admit the case of our A-torsors, i.e. let the following be the type of A-fibre bundles over X:

$$\sum_{T:X\to\mathcal{U}} \|T_x = A_x\|$$

Then, A-torsors will in particular also be A-fibre bundles.
A canonical trivialization for fibre bundles with constant prescribed fiber is given in ([7])[Definition 4.9, Definition 4.11] – but this works for the more general notion as well. The canonical trivialization is given by

$$V_T :\equiv \sum_{x:X} T_x = A_x$$

Then, from the definition above, we get that $\pi_1 : V_T \to X$ is surjective and the second projection will give a trivialization witness for the pullback of T along π_1. Now, to still work just with spectra over one fixed space, we have to push-forward the construction above. But that really just means we have to take a dependent product instead of a sum. So the resulting dependent group, were T is trivialized at $i = 1$, is just:

$$x \mapsto \prod_{T_x = *} A_{x,i}$$

This works also in higher degrees, for finitely many cohomology classes simultaneously and can be turned into an Eilenberg-MacLane-spectrum. All of this is the topic of lemma 3.3.

Definition 3.1
Let $k : \mathbb{N}$.

(a) A dependent function $\chi : (x : X) \to A_{x,k}$ is called a *((k − 1)-)gerbe*.

(b) A *(mere) resolution* of a gerbe $\chi : (x : X) \to A_{x,k}$ is a type M_χ together with a map $\iota : A \to M_\chi$, such that the induced map on gerbes

$$\iota_k : \prod_{x:X} A_{x,k} \to \prod_{x:X} M_{\chi,k}$$

maps χ to a class (merely) equal to $*$.

(c) A *resolving sequence* for $\chi : (x : X) \to A_{x,k}$ is a short exact sequence of Eilenberg-MacLane spectra over X, such that the first map resolves χ.

We will sometimes call these resolutions *local resolutions*, to emphasize the difference to classical resolutions, which usually resolve all cohomology classes at once.

It is possible to resolve finitely many cohomology classes for $k = 1$ using the construction in the proof of lemma 3.2 below.
Fortunately, it is possible to make this construction pointwise, which will be the following lemma and then extend it to Eilenberg-MacLane spectra over X, which will be the almost identical lemma 3.3.

Lemma 3.2
Let A be an Eilenberg-MacLane spectrum, and $\chi_1, \dots, \chi_n : A_{x,1}$.

(a) There is a morphism

$$\iota_{\chi_1,\dots,\chi_n} : A \to M_{\chi_1,\dots,\chi_n}$$

(b) The map ι is a monomorphism at degree 0.

Proof (a) Take

$$M_{(\chi_1,\dots,\chi_n),l} :\equiv \left(\prod_{i=1,\dots,n} (\chi_i = *) \right) \to A_l$$

and for $\iota_{(\chi_1,\dots,\chi_n),l}(a)$ the constant function with value $a : A_l$. Then, for any $i : \mathbb{N}$ there is

$$\mathrm{id}_{\chi_i = *} : \prod_{p:\chi_i = *} \chi_i = *$$

so we have $\iota_{\chi_1,\dots,\chi_n}(\chi_i) = *$ for any $1 \le i \le n$.

(b) A_1 is connected, so we merely have $p_i : \chi_i = *$ and therefore merely

$$p : \prod_{i=1}^{n} (\chi_i = *)$$

We want to show, that for all $a, a' : A_0$ we can conclude $a = a'$ from $\iota(a) = \iota(a')$. The latter is a proposition, so we can use p. But then we have:

$$\iota(a)(p) = \iota(a')(p)$$

and therefore, by definition of ι, $a = a'$.

Lemma 3.3
Let A be an Eilenberg-MacLane spectrum over X, $k : \mathbb{N}$ and $\chi_1, \ldots, \chi_n : (x : X) \to A_{x,k+1}$ finitely many cohomology classes.

(a) There is a morphism
$$\iota_{\chi_1,\ldots,\chi_n} : A \to M_{\chi_1,\ldots,\chi_n}$$
such that $(\iota_{(\chi_1,\ldots,\chi_n),k+1})(\chi_i) = *$.

(b) The maps ι and $\hat{\iota}$ are pointwise monomorphisms at degree 0.

Proof (a) By applying lemma 3.2 (a) pointwise and function extensionality.

(b) By applying lemma 3.2 (c) pointwise.

We can use lemma 3.3 (b) to construct short exact sequences, such that $\hat{\iota}^*(\chi_i)$ is zero, which will later help us to extend morphisms between long exact sequences. However, it is sometimes easier to deal with lemma 3.3 (a), so this will appear again as well.
It is natural and will later be useful, to compare different resolutions, sometimes of the same gerbe, sometimes of different gerbes. To do that, we can use morphisms:

Definition 3.4
A morphism of resolving sequences for $\chi : (x : X) \to A_{x,k}$ and $\xi : (x : X) \to A'_{x,k}$, is for all $x : X$ and $i \in \mathbb{N}$ a morphism of sequences of the following form:

$$\begin{array}{ccccc} A_{x,i} & \longrightarrow & R_{\chi,x,i} & \longrightarrow & C_{\chi,x,i} \\ \downarrow{\scriptstyle \varphi_{x,i}} & & \downarrow & & \downarrow \\ A'_{x,i} & \longrightarrow & R_{\xi,x,i} & \longrightarrow & C_{\xi,x,i} \end{array}$$

– that means all squares commute and $\varphi_{x,k}(\chi_x) = \xi_x$.

As the author was made aware by a comment of David Wärn, this definition lacks a natural coherence between the two triviality proofs of the cohomology classes. This seems to be no problem so far.
In most applications, φ will be the identity and we will use these morphisms to compare different resolutions of the same class. One exception is the following:

Lemma 3.5
Let $\chi : (x : X) \to A_{x,k}$ and $\varphi : A \to B$ be a morphisms of spectra over X, then there is a span of morphisms between the standard local resolution for $\varphi(\chi) :\equiv (x \mapsto \varphi_x(\chi_x))$ and the standard local resolution for χ, of the following shape:

$$\begin{array}{ccccc} A & \xrightarrow{\hat{\iota}_\chi} & \hat{M}_\chi & \longrightarrow & \mathrm{Cok}(\hat{\iota}_\chi) \\ \downarrow & & \downarrow & & \vdots \downarrow \\ B & \longrightarrow & ((\chi_x = *) \to B_x, k)_{x:X,k:\mathbb{N}} & \longrightarrow & \mathrm{Cok}(\ldots) \\ \| & & \uparrow & & \vdots \uparrow \\ B & \longrightarrow & \hat{M}_{\varphi(\chi)} & \longrightarrow & \mathrm{Cok}(\ldots) \end{array}$$

Proof For a morphism $\varphi : A \to B$ of spectra over X, we can construct a start of the desired morphism:

$$\begin{array}{ccc} B_{x,0} & \longrightarrow & M_{\varphi(\chi),x,0} \\ {\scriptstyle \varphi_{x,0}}\uparrow & & \uparrow \\ A_{x,0} & \longrightarrow & M_{\chi,x,0} \end{array}$$

where the new arrow is a morphism of groups, given by composition with the map $(\chi_x = *) \to (\varphi_x(\chi_x) = *)$ and $\varphi_{x,0}$. We can take connective covers and get an induced morphism between the cokernels:

$$\begin{array}{ccccc} B_x & \xrightarrow{\hat{\iota}_{\varphi(\chi)}} & \hat{M}_{\varphi(\chi),x} & \longrightarrow & \mathrm{Cok}(\hat{\iota}_{\varphi(\chi)}) \\ {\scriptstyle \varphi_x}\uparrow & & \uparrow & & \vdots\uparrow \\ A_x & \xrightarrow{\hat{\iota}_\chi} & \hat{M}_{\chi,x} & \longrightarrow & \mathrm{Cok}(\hat{\iota}) \end{array}$$

Lemma 3.6
Let $\chi_1, \ldots, \chi_n : (x : X) \to A_{x,k}$, then there is a morphism from the standard sequence for each χ_i to the sequence for all of $\chi_1, \ldots, \chi_n$.

Proof Same construction as above, this time using precomposition with (and no postcomposition)

$$\left(\prod_{i=1}^{n} (\chi_{i,x} = *) \right) \to (\chi_{i,x} = *)$$

4 ∂-Functors

The following definition, from ([3, p. 2.1]) and originally from ([4]), is specialized to our needs. Grothendieck makes a definition for additive functors from an abelian category to a preadditive category. We just state it for abelian categories and will later apply it only to functors from dependent abelian groups over a fixed type to abelian groups. While we expect both of these types to admit the structure of an abelian category, we have proved neither and we will use neither in any proofs in this article. Instead, we will only use basic facts about abelian groups.

Definition 4.1
Let C and C' be abelian categories.
A *∂-Functor* is a collection of functors $T^i : C \to C'$, where $0 \leq i < a$ with $a \in \mathbb{N} \cup \{\infty\}$, together with a collection of connecting morphisms $\partial_{S,i}$ for any short exact sequence S and $0 \leq i \leq a$, subject to the following conditions:

(a) Let S be a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in C. Applying the T^i yields a complex, together with connecting morphisms $(\partial_{S,i})_{i\in\mathbb{N}}$:

$$T^0(A') \longrightarrow T^0(A) \longrightarrow T^0(A'') \xrightarrow{\partial_{S,0}} T^1(A') \longrightarrow T^1(A) \longrightarrow \dots$$

(b) For any homomorphism to a second short exact sequence

$$0 \to B' \to B \to B'' \to 0$$

and any valid i the corresponding square commutes:

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\partial} & T^{i+1}(A') \\ \downarrow & & \downarrow \\ T^i(B'') & \xrightarrow{\partial} & T^{i+1}(B') \end{array}$$

Definition 4.2
Let T and T' be ∂-Functors defined for the same indices.
A *morphism of ∂-Functors* $f : T \to T'$ is given by a natural transformation $f^i : T^i \to T'^i$ for each valid i, such that for any short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

the following square commutes:

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\partial} & T^{i+1}(A') \\ \downarrow f^i_{A''} & & \downarrow f^{i+1}_{A'} \\ T'^i(A'') & \xrightarrow{\partial} & T'^{i+1}(A') \end{array}$$

Definition 4.3
A ∂-Functor T is called *universal*, if for any T', defined for the same indices, any natural transformation $f^0 : T^0 \to T'^0$ extends uniquely to a morphism of ∂-Functors $f : T \to T'$.

The following is provable by a constructive adaption of Prop 2.2.1 in the Tohoku-Paper:

Theorem 4.4
Let X be a type. The functors H^i mapping dependent abelian groups over X to abelian groups form a universal ∂-Functor.

Before proving the theorem, we will establish some lemmas about extending morphisms between ∂-Functors. In the following, let X be a type, A a dependent abelian group over X, T a ∂-Functor and f a morphism of ∂-Functors, defined up to degree $i-1$.

Lemma 4.5
For any gerbe $\chi : (x : X) \to A_{x,k}$ let S be a short exact sequence

$$A \xrightarrow{\iota_\chi} R_\chi \xrightarrow{c_\chi} C_\chi$$

of Eilenberg-MacLane spectra over X resolving χ. There is a unique

$$\mathrm{ext}(\chi, S) : T^i(A)$$

(with T as in theorem 4.4) such that for any $x : H^{i-1}(C_\chi)$ with $\partial_{H,S,i-1}(x) = |\chi|$ we have $\partial_{T,S,i-1}(f^{i-1}(x)) = \mathrm{ext}(\chi, S)$.

Proof (of lemma 4.5) Let

$$A \to R_\chi \to C_\chi$$

be a short exact sequence of Eilenberg-MacLane spectra resolving χ. The following diagram commutes:

$$\begin{array}{ccccccccc}
H^{i-1}(A) & \longrightarrow & H^{i-1}(R_\chi) & \longrightarrow & H^{i-1}(C_\chi) & \xrightarrow{\partial} & H^i(A) & \xrightarrow{\iota_\chi^*} & H^i(R_\chi)\dots \\
\downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^{i-1}} & & & & \\
T^{i-1}(A) & \longrightarrow & T^{i-1}(R_\chi) & \xrightarrow{c_\chi^*} & T^{i-1}(C_\chi) & \xrightarrow{\partial} & T^i(A) & \longrightarrow & T^i(R_\chi)\dots
\end{array}$$

The upper row is exact and the lower row is a complex.
Let $E(\chi, S)$ be the type of all possible values of f^i in $T^i(A)$, with which we mean all $y : T^i(A)$ such that there merely is $x : H^{i-1}(C_\chi)$ with $\partial(x) = |\chi|$ and $\partial(f^{i-1}(x)) = y$. Then $E(\chi, S)$ is inhabited, since $\iota_\chi(|\chi|) = 0$ and by exactness, there has to be a mere preimage under ∂. So we need to show, that $E(\chi, S)$ is a proposition.
Let $x : H^{i-1}(C_\chi)$ such that $\partial(x) = |\chi|$. Then any other element with this property will be of the form $x + k$, with k in the kernel of ∂. Any k like that, has a mere preimage $k' : H^{i-1}(R_\chi)$ and since the lower row is a complex, we have $\partial(c_\chi^*(f^{i-1}(k'))) = 0$.
So for any extension $y : T^i(A)$ we have

$$\begin{aligned}
y &= \partial(f^{i-1}(x+k)) \\
&= \partial(f^{i-1}(x)) + \partial(f^{i-1}(k)) \\
&= \partial(f^{i-1}(x)) + \partial(c_\chi^*(f^{i-1}(k'))) \\
&= \partial(f^{i-1}(x))
\end{aligned}$$

This means we can define $\mathrm{ext}(\chi, S)$ to be the unique element of $E(\chi, S)$.

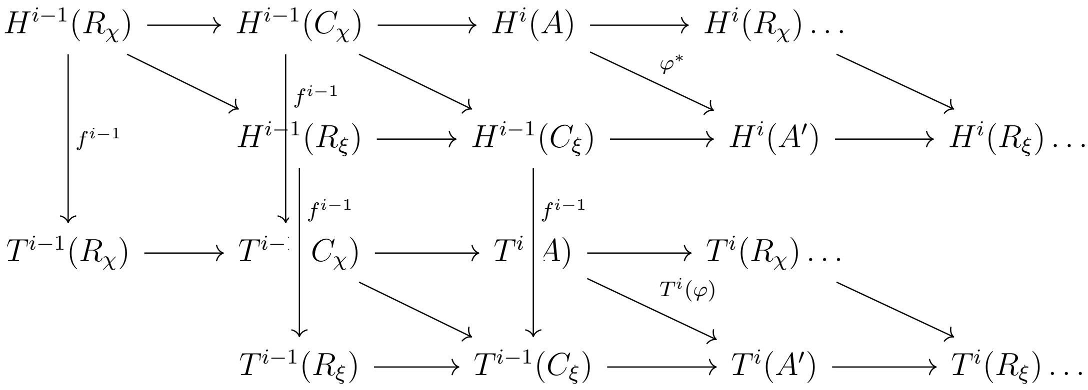
Lemma 4.6
For any cohomology classes $\chi : (x : X) \to A_{x,k}$, $\xi : (x : X) \to A_{x,k}$ and any morphism

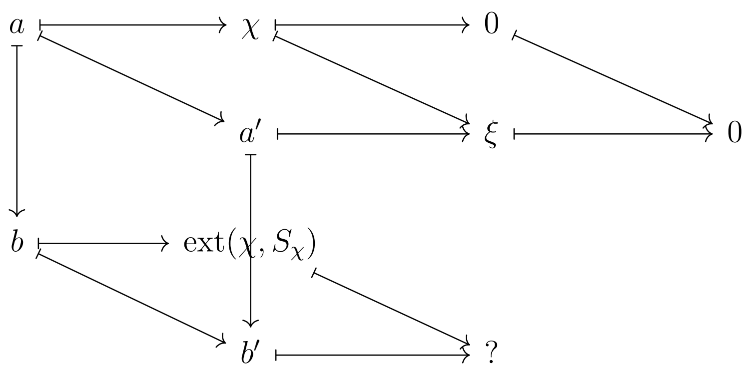
$$\begin{array}{ccccc} A & \longrightarrow & R_\chi & \longrightarrow & C_\chi \\ \downarrow{\scriptstyle\varphi} & & \downarrow & & \downarrow \\ A' & \longrightarrow & R_\xi & \longrightarrow & C_\xi \end{array}$$

of short exact resolving sequences S_χ and S_ξ for χ and ξ in the sense of definition 3.4, we have:

$$T^k(\varphi)(\mathrm{ext}(\chi, S_\chi)) = \mathrm{ext}(\xi, S_\xi)$$

Proof (of lemma 4.6) Apply the ∂-Functors H and T to the morphism of resolving sequences, to get the following diagram:





From exactness of the upper sequence, we get that there is a preimage a of χ. Let a' denote the image of a in $H^{i-1}(C_\xi)$, then a' will be a preimage of ξ in the other sequence by commutativity. That means, b' will be mapped to $\mathrm{ext}(\xi, S_\xi)$, but by commutativity, $\mathrm{ext}(\chi, S_\chi)$ will be mapped to the same thing by $T^i(\varphi)$. So

$$T^i(\varphi)(\mathrm{ext}(\chi, S_\chi)) = \mathrm{ext}(\xi, S_\xi)$$

Proof (of theorem 4.4) First of all, H is a ∂-functor by **??**.
To extend a given morphism f^0, we will construct $f^i : H^i \to T^i$ recursively for $i \in \mathbb{N}$. The construction of f^i will be done pointwise, for each element of $H^i(X, A)$

using the recursion principle for 0-truncation. Since the codomain of f^i is 0-truncated, the latter means, we can just assume that each element is of the form $|\chi| : H^i(X, A)$ for some cohomology class $\chi : (x : X) \to A_{x,i}$.
So assume $i \neq 0$ and let $\chi : (x : X) \to A_{x,i}$. By lemma 3.3 we have the standard short exact sequence S_χ resolving χ, so we can use lemma 4.5 to construct an image $f^i(|\chi|) :\equiv \mathrm{ext}(\chi, S_\chi)$.
To see that a natural homomorphism of abelian groups f^i can be constructed in this way, we will lemma 4.6.
We can apply lemma 4.6 to the morphism from lemma 3.6, to see that we could have define $f^i(|\chi|)$ as well by a resolution of the three classes $\chi, \xi, \chi + \xi : (x : X) \to A_{x,i}$. But this is already enough to conclude that $f^i(|\chi| + |\xi|) = f^i(|\chi) + f^i(\xi)$, by the homomorphism properties of the maps involved in the construction in lemma 4.5.
For a morphism $\varphi : A \to A'$ of spectra over X, we can use the morphism of resolving sequences from lemma 3.5. Then lemma 4.6 tells us directly, that f^i is natural.
For the commutativity of f^i with connecting morphisms, let $x : X$ and consider a short exact sequence S:

$$A_x \xrightarrow{\psi_x} A'_x \longrightarrow A''_x$$

For any $\chi_x : A_{x,i}$, we can construct the following diagram:

$$\begin{array}{ccccc}
A_{x,i} & \xrightarrow{\psi_{x,i}} & A'_{x,i} & \longrightarrow & A''_{x,i} \\
\| & & \downarrow & & \downarrow \\
A_{x,i} & \xrightarrow{\widehat{\mathrm{const} \circ \psi_{x,i}}} & \widehat{\big((\chi_x = *) \to A'_{x,i}\big)} & \longrightarrow & \mathrm{Cok}(\widehat{\mathrm{const} \circ \psi_{x,i}}) \\
\| & & \uparrow & & \uparrow \\
A_{x,i} & \xrightarrow{\widehat{\mathrm{const}}} & \widehat{((\chi_x = *) \to A_{x,i})} & \longrightarrow & \mathrm{Cok}(\widehat{\mathrm{const}})
\end{array}$$

If we know $\psi_{x,i}(\chi_x) = *$, then $\chi_{x,i}$ will also equal $*$ in $\big((\chi_x = *) \to A'_{x,i}\big)$ and $((\chi_x = *) \to A_{x,i})$. This is still true in connective covers. So we have morphisms between three resolving sequences and we can apply lemma 4.6 twice to get:

$$\mathrm{ext}(\chi, S) = \mathrm{ext}(\chi, S_\chi) \equiv f^i(|\chi|)$$

– where S is the given short exact sequence and S_χ the standard resolving sequence for χ.
What remains now, is to use this to show that the following diagram (from the LES for S) commutes:

$$\begin{array}{ccc}
H^{i-1}(A'') & \xrightarrow{\partial} & H^i(A) \\
\downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} \\
T^{i-1}(A'') & \xrightarrow{\partial} & T^i(A)
\end{array}$$

So let $|\xi| : H^{i-1}(A'')$ and $|\chi| :\equiv \partial(|\xi|)$. Then, by exactness, we know $\psi^*(|\chi|) : H^i(A')$ is zero. This means the underlying class is merely zero and we can apply what we just proved, to (merely) get that

$$\partial(f^{i-1}(|\xi|)) = \mathrm{ext}(\chi, S) = \mathrm{ext}(\chi, S_\chi) \equiv f^i(|\chi|) = f^i(\partial(|\xi|))$$

Done.

5 Higher direct images

This is a guess I haven't really thought about.

Definition 5.1
Let $f : X \to Y$ be a map.

(a) Let A be an abelian group over X. Then

$$\left(f_* A\right)_y :\equiv \prod_{x:\mathrm{fib}_f(y)} A_x$$

is the *direct image* of A.

(b) Let A be a spectrum over X. Then

$$\left(f_* A\right)_{y,k} :\equiv \prod_{x:\mathrm{fib}_f(y)} A_{x,k}$$

is a spectrum over Y and the following groups over Y

$$\left(R^k f_* A\right)_y :\equiv \|f_* A_{y,k}\|_0$$

are the higher direct image groups of f and A over Y.

Theorem 5.2
Let $f : X \to Y$ be a map. Then, for varying $k \in \mathbb{N}$, the functors $R^k f_*$ form a universal ∂-functor.

Proof Apply theorem 4.4 for each $y : Y$.

Definition 5.3
Let $f : X \to Y$ be a map. For A an abelian group over Y, the *pull-back* of A along f is the dependent abelian group given for $x : X$ by

$$\left(f^* A\right)_x :\equiv A_{f(x)}$$

Remark 5.4
Let the following be a pullback square of types

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

and A a spectrum over X. There is a morphism

$$g^* R^i f_* A \to R^i f'_*(g'^* A)$$

Proof Right side:

$$\begin{aligned} R^i f'_*(g'^* A) &\equiv R^i f'_*(x' \mapsto A_{g'(x')}) \\ &\equiv s' \mapsto \left\| \prod_{x' : \mathrm{fib}_{f'(s')}} A_{g'(x'),i} \right\|_0 \end{aligned}$$

Left side:

$$\begin{aligned} g^* R^i f_* A &\equiv g^* \left(s \mapsto \left\| \prod_{x : \mathrm{fib}_f(s)} A_{x,i} \right\|_0 \right) \\ &\equiv s' \mapsto \left\| \prod_{x : \mathrm{fib}_f(g(s'))} A_{x,i} \right\|_0 \end{aligned}$$

Use the general fact about pullbacks, that

$$\prod_{s' : S'} \mathrm{fib}_{f'}(s') \simeq \mathrm{fib}_f(g(s'))$$

is induced by g'.

This is weird - the induced map shouldn't be an equivalence in general.

6 Local Resolutions in Synthetic Algebraic Geometry

In this section we construct local resolutions for a good subcategory of the depedent modules over a scheme in synthetic algebraic geometry.
Let R be a fixed commutative ring, serving as a base ring for the definitions from the preprint [2], we will now import:

Definition 6.1
Let A be an R-algebra.

(a) For $f : A$ let

$$D(f) :\equiv f \text{ is invertible}$$

be the proposition that f has a multiplicative inverse.

(b) A subtype $U : X \to \mathrm{Prop}$ of any type X is open, if for all $x : X$, there merely are $f_1, \ldots, f_n$ such that $U(x) = D(f_1) \vee \cdots \vee D(f_n)$.

(c) The type

$$\mathrm{Spec}A :\equiv \mathrm{Hom}_R(A, R)$$

of R-algebra homomorphisms is called the *spectrum of A* and there is a correspondence with external affine spectra in the Zariski-topos.

(d) A scheme is a type X which is covered by finitely many open affine subtypes. These schemes are expected to correspond to external quasi compact, quasi separated schemes, locally of finite type.

Definition 6.2
Let M be an R-module. M is *weakly quasi coherent*, if the canonical R-linear map

$$\frac{m}{f^k} \mapsto ((\_ : f \text{ inv}) \mapsto f^{-k}\frac{m}{f^k}) : M_f \to M^{D(f)}$$

is an isomorphism. A dependent R-module $M : X \to R\text{-Mod}$ is weakly quasi coherent, if it is pointwise weakly quasi coherent.

Conjecture 6.3
Let X be an affine scheme and $M : X \to R\text{-Mod}$ weakly quasi coherent, then for all $n > 0$:

$$H^n(X, M) = 0.$$

Proof The case $n = 1$ is proven in [2] and the method there extends at least to arbitrary external n.

Definition 6.4
Let X, Y be schemes and.

(a) For $M : Y \to R\text{-Mod}$ and $f : X \to Y$ let $f^*M :\equiv (x : X) \mapsto M_{f(x)}$.

(b) For $M : X \to R\text{-Mod}$ and $f : X \to Y$ let $f_*M :\equiv (y : Y) \mapsto \prod_{x:\mathrm{fib}_f(y)} M_{\pi_1(x)}$.

Both operations preserve weakly quasi coherent modules by [2][Theorem 9.1.11]. At the heart of the construction of local resolutions below, is what is called *Zariski-local choice* in [2] and justified as an axiom there. A special case is local-triviality of gerbes:

Axiom 6.5
Let $X = \mathrm{Spec}A$ be an affine scheme and $M : X \to R\text{-Mod}$. For $n > 0$ and each $G : \prod_{x:X} K(M_x, n)$ there merely is a Zariski cover, i.e. coprime $f_1, \ldots, f_n : A$ such that on each subtype $D(f_i) :\equiv (x : X) \mapsto D(f_i(x))$, G is trivial.

With that, there merely are finite covers of schemes trivializing any M-gerbe G. Let us denote the coproduct of such a cover with U_G, so we have a surjection $u_G : U_G \to X$ such that $\prod_{x:U_G} G_x = M_x$. If we started with a weakly quasi coherent M, we get an injective map of weakly quasi coherent modules:

$$M_x \to M_x^{\mathrm{fib}_{u_G}(x)}$$

where the domain is weakly quasi coherent, since it is $(u_{G*}u_G^* M)_x$.
To get a resolution from that, we need to see that cokernels of monomorphisms of weakly quasi coherent modules are weakly quasi coherent. In symbols, for $N \subseteq M$ one of those morphisms, we need:

$$(M/N)_f = (M/N)^{D(f)}.$$

by algebra $(M/N)_f = M_f/N_f$. This means we are done, if $(M/N)^{D(f)} = M^{D(f)}/N^{D(f)}$. To see this holds, let us consider $0 \to N \to M \to M/N \to 0$ as a short exact sequence of dependent modules, over the subtype of the point $D(f) \subseteq 1 = \mathrm{Spec}R$. Then, taking global sections, we have an exact sequence

$$0 \to N^{D(f)} \to M^{D(f)} \to (M/N)^{D(f)} \to H^1(D(f), M)$$

– but $D(f) = \mathrm{Spec}R_f$ is affine, so the last term is 0 and $(M/N)^{D(f)}$ is the cokernel $M^{D(f)}/N^{D(f)}$. So we know that dependent wqc modules over a scheme, always merely have local resolutions consisting of dependent wqc modules.
By **??** 6.3, the types $K(M_x, n)^{U_\chi}$, where M_x is wqc and $n > 0$ are 0-connected – note that M_x is weakly quasi coherent on U_χ, since it is the push-forwad of M_x along the map $U_\chi \to 1$.
This means that $\left(K(M_x, n)^{U_\chi}\right)_{n:\mathbb{N}}$ is an Eilenberg-MacLane spectrum. So we have found resolving sequences for all gerbes on a scheme with coefficients in weakly quasic coherent modules.

Appendix

Here, we will prove things about a general modality.

Definition 6.6
A modality (on a universe U) is a map $\bigcirc : U \to U$ together with a dependent map $\eta : \prod_{X:U} X \to \bigcirc X$ such that for all $A : U$ and $B : \bigcirc A \to U$ the following map is an equivalence:

$$\_ \circ \eta_A : \left(\prod_{x:\bigcirc A} \bigcirc B(x)\right) \to \left(\prod_{y:A} \bigcirc B(\eta_A(y))\right)$$

Remark 6.7
Let $f : X \to Y$ be any map. Then there is a map $\bigcirc f : \bigcirc X \to \bigcirc Y$, constructed by using the property above with $A \equiv X$ and $B \equiv \_ \mapsto Y$. This is natural in that the following square commutes:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \bigcirc X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bigcirc f} \\ Y & \xrightarrow{\eta_Y} & \bigcirc Y \end{array}$$

We will drop the index of η from now on, and, for example, just write $\eta : X \to \bigcirc X$. For any pointed X, the type $\bigcirc X$ is pointed by $* :\equiv \eta(*)$.

Definition 6.8

(a) A type X is $\bigcirc$-connected, if $\bigcirc X = 1$.

(b) In the case $\bigcirc \equiv \|\_\|_k$, we also just say "k-connected".

Definition 6.9

Let X be a pointed type.

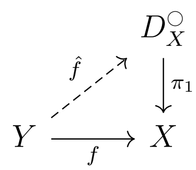
(a) The $\bigcirc$-*connected cover* of X is the type:

$$D_X^{\bigcirc} :\equiv \sum_{x:X} \eta(x) = \eta(*)$$

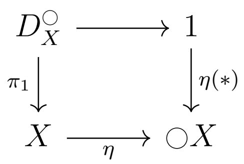
(b) In the case $\bigcirc \equiv \|\_\|_k$, we write also just D_X^k.

Lemma 6.10

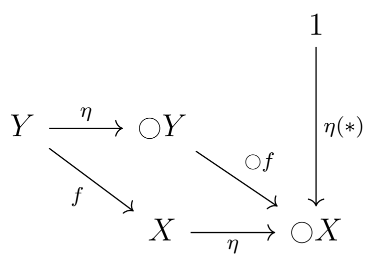
Let X and Y be pointed types. If Y is $\bigcirc$-connected, then any pointed map $f : Y \to X$ lifts to a map $\hat{f} : Y \to D_X^{\bigcirc}$, i.e. the following diagram commutes:



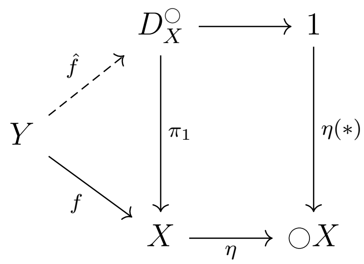
Proof We will use the universal property of the following pullback square to construct $\hat{f}$:



To start with the construction of a cone, we apply η to the given pointed map $f : Y \to X$:



There is always a map $\bigcirc Y \to 1$ and since $\bigcirc Y = 1$, it is enough to show that $\bigcirc f(*) = \eta(*)$ holds, to produce a cone. Since f is pointed, we have $f(*) = *$ and therefore $\eta(f(*)) = \eta(*)$. By naturality of η, that yields $\bigcirc f(*) \equiv \bigcirc f(\eta(*)) = \eta(*)$. So we can complete the diagram above to a cone and get the desired induced map:



It turns out, that we can lift equalities along π_1 in the following sense:

Corollary 6.11
In the situation of lemma 6.10, if we have $y : Y$ and $f(y) = f(*)$, then $\hat{f}(y) = \hat{f}(*)$.

Proof The lemma provides us with a map from pointed maps $Y \to X$ to lifts $Y \to D_X^{\bigcirc}$. This means we are done, if we manage to reformulate $f(y) = f(*)$ as an equality of pointed maps $1 \to X$.
So let $p : f(*) = *$ be the pointing of f and $q : f(y) = f(*)$ be the given equality. Then $(\_ \mapsto f(*)) : 1 \to X$ is pointed by p and $(\_ \mapsto f(y)) : 1 \to X$ is pointed by $q \cdot p$. But the requirement for maps equal maps $\varphi = \psi$ to equal as pointed maps, is just that the induced $\varphi(*) = \psi(*)$ is compatible with the pointings of φ and ψ, which is witnessed by $\mathrm{refl}_{q \cdot p}$ in our case.

Remark 6.12
If, in the situation of the lemma, the pointed map $f : Y \to X$ is an monomorphism, then $\hat{f}$ is also an monomorphism.

Proof Let $x, y : Y$ and $p : \hat{f}(x) = \hat{f}(y)$. So $\pi_1(\hat{f}(x)) = \pi_1(\hat{f}(y))$ and $f(x) = f(y)$ because of commutativity. Then, $x = y$ because f is an monomorphism.

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